# Critical Coarsening without Surface Tension: The Universality Class of the Voter Model

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We show that the two-dimensional voter model, usually considered to be only a marginal coarsening system, represents a broad class of models for which phase ordering takes place without surface tension. We argue that voter-like growth is generically observed at order-disorder nonequilibrium transitions solely driven by interfacial noise between dynamically symmetric absorbing states.

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Coarsening phenomena occur in a large variety of situations in and out of physics, ranging from the spinodal decomposition of alloys [1] to population dynamics [2]. At a fundamental level, phase ordering challenges our capacity to deal with nonequilibrium systems and our understanding of the mechanisms determining different universality classes. In many cases, phase competition is driven by surface tension, leading to "curvature-driven" growth. Coarsening patterns are then characterized by a single length scale $L(t) \sim t^{1/z}$, with $z$ depending only on general symmetry and conservation properties of the system [1]. For instance, $z = 2$ for the common case of a nonconserved scalar order parameter, a large class including the Ising model quenched off criticality. In this context, the two-dimensional voter model, a caricatural process in which sites on a square lattice adopt the opinion of a randomly chosen neighbor, stands out as an exception. Its coarsening process, which gives rise to patterns with clusters of all sizes between 1 and $\sqrt{t}$ (Fig. 1) [3,4], is characterized by a slow, logarithmic decay of the density of interfaces $\rho \sim 1/\ln t$ [as opposed to the algebraic decay $\rho \sim 1/L(t) \sim t^{-1/z}$ of curvature-driven growth]. The marginality of the voter model is usually attributed to the exceptional character of its analytic properties [3–7].

In this Letter, we show that, in fact, large classes of models exhibit the same type of domain growth as the voter model, without being endowed with any of its peculiar symmetry and integrability properties. We argue that voter-like coarsening is best defined by the absence of surface tension and that it is generically observed at the transitions between disordered and fully ordered phases in the absence of bulk fluctuations, when these nonequilibrium transitions are driven by interfacial noise only. Finally, we discuss the universality of the scaling properties associated with voter-like critical points.

We first review the properties—in any space dimension $d$—of the voter model (VM), emphasizing those of importance for defining generalized models. A "voter" (or spin) residing on site $\mathbf{x}$ of a hypercubic lattice can have two different opinions $s_{\mathbf{x}} = \pm 1$. An elementary move consists in randomly choosing one site and assigning to it the opinion of one of its randomly chosen nearest neighbors (nn). Thus the two homogeneous configurations (where all spins are either + or −) are absorbing states, and the model is $Z_2$ symmetric under global inversion ($s_{\mathbf{x}} \to -s_{\mathbf{x}}$). Recasting the dynamic rule in terms of nn pair updating, a pair of opposite spins $+-$ is randomly selected, and evolves to a $++$ or $--$ pair with equal (1/2) probabilities. The rates of creation of + and − spins being equal, any initial value of the global magnetization $m$ is conserved in the limit of large system sizes.

Another prominent feature of the VM is a "duality" [3,4] with a system of coalescing random walks: going backward in time, the successive ancestors of a given spin follow the trail of a simple random walk (RW); comparing the values of several spins shows that their associated RWs necessarily merge upon encounter. This correspondence allows one to solve many aspects of the kinetics of the VM, because it implies that the correlation functions between an arbitrary number of spins form a closed hierarchy of diffusion equations [4–6]. In particular, the calculation of the density of interfaces $\rho_m(t)$ (i.e., the fraction of $+-$ nn pairs), starting from random initial conditions (ric) of magnetization $m$, is ultimately given by the probability that a RW, initially at unit distance from the origin, has not yet reached it at time $t$. Therefore, owing to the recurrence properties of RWs, the VM shows coarsening for

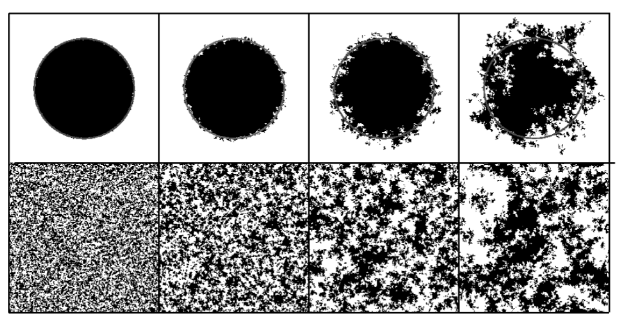

FIG. 1. Illustration of the domain growth in the $d = 2$ VM (system size $256^2$). Top: Snapshots at times $t = 4, 16, 64, 256$ during the evolution of a bubble of initial radius $r_0 = 180$ (thin circle). Bottom: same from symmetric ric.

  

$d \leq 2$ [i.e., $\rho_m(t) \to 0$ when $t \to \infty$]. For the "marginal" case $d = 2$—on which we mainly focus henceforth—one finds the slow logarithmic decay [4–6]

$$\rho_m(t) \propto (1 - m^2)\left[\frac{2\pi D}{\ln t} + \mathcal{O}\left(\frac{1}{\ln^2 t}\right)\right], \qquad (1)$$

$D$ (= 1/4 in the nn case) being the diffusion constant of the underlying RW [8]. This peculiar behavior, contrasting with the algebraic decay encountered in curvature-driven growth, thus seems tantamount to the fact that $d = 2$ is the critical dimension for the recurrence of RWs.

Beyond these analytic properties, the nature of the coarsening displayed by the VM in two dimensions is highlighted by studying the evolution of a "bubble" of one phase embedded into another. For nonconserved curvature-driven growth, the volume of a bubble decreases linearly with time under the effect of surface tension. In the VM, however, large bubbles do not shrink but slowly disintegrate as their boundary roughens diffusively to reach a typical width of the order of their initial radius $r_0$ (Fig. 1, top). As long as $\sqrt{t} \ll r_0$, conservation of the global magnetization is still effective, and implies that radially averaged magnetization profiles $m(r,t)$ have a stationary middle point. This indicates that curvature plays no role, a fact further confirmed by the observation that the $m(r,t)$ curves are the same as the transverse profiles of an infinite, straight interface.

We interpret the above behavior as characteristic of the absence of surface tension, a "physical" property which, together with the fact that coarsening does occur with a logarithmic decay for $\rho_m(t)$, we conjecture to be constitutive of $d = 2$ voter-like domain growth. To investigate which of the properties outlined above (existence of two absorbing states, $Z_2$ symmetry, integrability via coalescing RWs, $m$ conservation) is necessary to produce voter-like growth, we now embed the two-dimensional VM described above into large families of rules.

Consider first that a $+-$ pair is now updated only with probability $p$ [$\Pr(+- \to +-) = 1 - p$], and that $\Pr(+- \to ++) = pq$ and $\Pr(+- \to --) = p(1-q)$, so that for $q \neq \frac{1}{2}$ the $m$ conservation is broken. Suppose next that $p$ and $q$ depend on the local configuration around the $+-$ pair, for instance via $n^+$ and $n^-$, the numbers of $+$ ($-$) nn of the $-$ ($+$) site in the central pair. Conservation of $m$ [$\Pr(+- \to ++) = \Pr(+- \to --)$] is then expressed by the condition $q_{n^+,n^-} = \frac{1}{2}$ (to be obeyed $\forall\, n^+, n^- \in \{1,2,3,4\}$), whereas $p_{n^-,n^+} = p_{n^+,n^-}$ and $q_{n^-,n^+} = 1 - q_{n^+,n^-}$ enforce $Z_2$ symmetry. As soon as the transition probabilities vary with the local configuration, the duality with coalescing RWs, used above to derive, e.g., Eq. (1), breaks down. This applies, with the exception of the usual VM ($p = 1, q = 1/2$), to any of the $m$-conserving and $Z_2$-symmetric rules ($q_{n^+,n^-} = 1/2$ and $p_{n^-,n^+} = p_{n^+,n^-}$). Nevertheless, our coarsening, bubble, and line experiments show [9] that such rules *do* behave like the simple VM, but with a diffusion constant $D \neq \frac{1}{4}$. Thus strict integrability is not a necessary condition for voter-like growth.

The subset of $m$-conserving rules without $Z_2$ symmetry is also of particular interest: no surface tension is expected, but one may wonder about the effects of the asymmetry on growth properties. Below, we use the case $p_{n^+,n^-} = 1/n^+$, which is nothing but the interface growth model introduced by Kaya, Kabakçıoğlu, and Erzan (KKE) of [10] right at its "delocalization" transition. The asymmetry of this rule manifests itself even at the macroscopic level, by looking at the evolution of the magnetization profiles in "line" or bubble experiments. As for the usual VM, the profiles obtained in both cases are identical, their widths scale as $(2D_{\mathrm{KKE}}t)^{1/2}$ with $D_{\mathrm{KKE}} = 0.36(2)$ [11], but they are asymmetric (Fig. 2a). Voter-like behavior is further confirmed by the study of phase ordering following ric of magnetization $m$. The density $\rho_m(t)$ displays the signature scaling spelled out by Eq. (1) (Fig. 2b). By measuring the autocorrelation $A(t)$ of a spin with itself in the initial condition, which (for $m = 0$) should scale [7] as $A(t) \approx (4\pi D t)^{-1}$, we find [9] that $D = 0.37(1)$—in nice agreement with the value $D_{\mathrm{KKE}}$ determined above—suggesting that the KKE rule asymptotically behaves like the usual VM with an *effective* diffusion constant.

All the other $m$-conserving, asymmetric rules that we have studied reveal a similar voter-like behavior. We conjecture that they almost all do so, as long as the $p_{n^+,n^-}$ are all strictly positive (to avoid possible "blocking" in certain configurations). In other words, $m$ conservation appears as a sufficient condition for voter-type growth, even in the absence of $Z_2$ symmetry or integrability. We now show that $m$ conservation is not a necessary condition since some nonconserving $Z_2$-symmetric rules with two absorbing states exhibit voter-like features.

If pair-update rules are convenient to control $m$ conservation, single-site update models suffice to study rules which need not possess this property. We thus consider a family of "kinetic Ising models," in which a spin is flipped

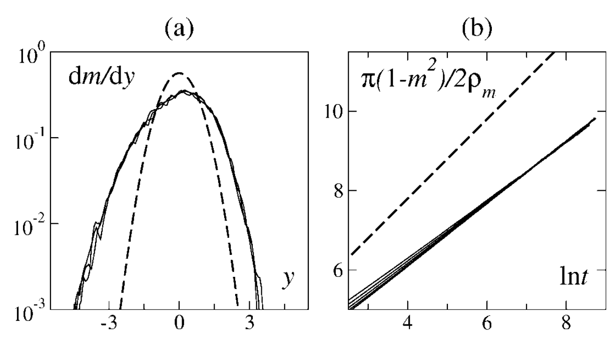


FIG. 2. Domain growth in the KKE model and the nn VM. (a) Derivative of rescaled magnetization profiles at times $t = 16$, 64, 256 [data from a bubble experiment; initial conditions: $m(r,0) = \mathrm{sgn}(r - r_0)$ with $r_0 = 4096$; $y = (r - r_0)/\sqrt{t}$]. Dashed line: exact result (see [10]) for the nn VM. (b) Scaling of $\rho_m(t)$ [as suggested by Eq. (1)] in coarsening experiments for $m = 0$, $\pm 0.2$, $\pm 0.4$ in systems of size $16\,384^2$; dashed line: exact result [Eq. (1) with $D = 1/4$] for the nn VM.

with a probability $r_{s,h}$ depending on its value $s = \pm 1$ and on its local field $h \in \{-4, -2, 0, 2, 4\}$, i.e., the sum of the values of its four nn. The $Z_2$-symmetric rules of this family, obeying $r_{-s,-h} = r_{s,h}$, can be parametrized by five transition probabilities $r_h \equiv r_{1,h}$. Such rules are usually not integrable. Again, the VM, given by $r_h = \frac{1}{2} - \frac{h}{8}$, is an exception. Its marginal situation in this context has already been noted [12,13] in the particular case where the transition probabilities depend only on $h$, a condition expressed here by $r_{-s,h} = 1 - r_{s,h}$. One is then left with only two free parameters, $p_b \equiv r_4$, $p_i \equiv r_2$, which can be, respectively, interpreted [13] as a measure of the strength of "bulk" and "interfacial" noise. In the $(p_i, p_b)$ plane, the VM $(p_i = \frac{1}{4}, p_b = 0)$ turns out [12,13] to be the end point of a line of Ising-like (second-order) phase transitions, at the border between the ordered and disordered phases in the absence of bulk noise $(p_b = 0)$.

We claim that this situation is generic: the four-parameter space of the general kinetic Ising model above without bulk noise (space defined by $r_4 = 0$) is divided into a disordered and an ordered region separated by a codimension-1 *critical* manifold of voter-like rules [14]. In Fig. 3, we show a section of this manifold in a plane which contains the VM. Its location was determined by studying the evolution of $\rho$ following ric with zero magnetization at various parameter values. Crossing the critical manifold, the characteristic logarithmic decay $\rho \propto 1/\ln t$ separates the region where $\rho \propto 1/\sqrt{t}$ (curvature-driven growth typical of the ordered phase) from the disordered region where $\rho$ saturates to finite values (Fig. 3b). A peculiar feature of this voter-like critical manifold is that, using the language of the renormalization group (RG), $m = 0$ is a "weakly attracting" fixed point of the dynamics: starting from any $m_0 \neq 0, \pm 1$ $[m_0 = m(0)]$, one observes that $m(t) \propto 1/\ln t$ (Fig. 3c). Yet, the decay of the interface density behaves as in the usual VM (Fig. 3d), even for $m_0 \neq 0$, because the slow evolution of $m(t)$ introduces only $\mathcal{O}(1/\ln^2 t)$ corrections to Eq. (1) and therefore does not alter the value of the effective diffusion constant. This also explains why our bubble and line experiments—which strictly speaking probe only a zero-measure set of random initial conditions—performed for rules on the voter-like manifold do give the same results as for the usual VM [9], in spite of the absence of $m$ conservation.

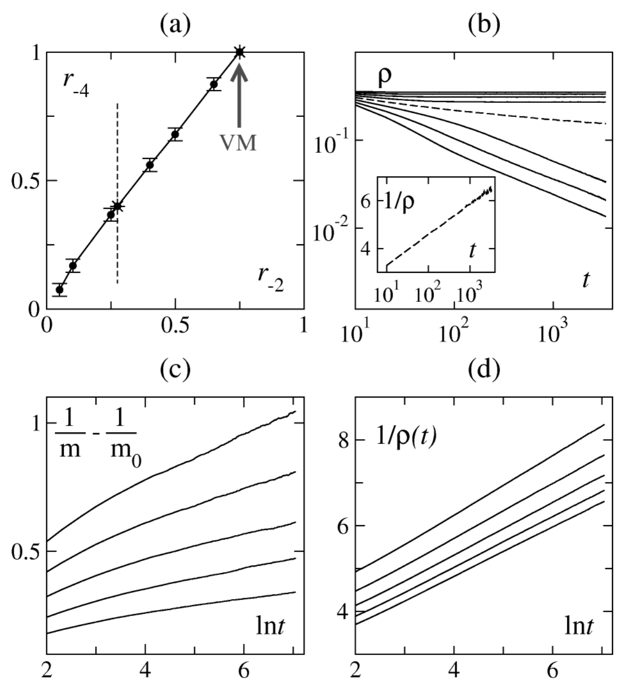


FIG. 3. Voter-like rules not conserving $m$: (a) phase diagram in the plane $(r_{-2}, r_{-4})$ for $r_0 = \frac{1}{2}, r_2 = r_{-4}/4$. (b) $\rho(t)$ from symmetric ric increasing $r_{-4}$ (from top to bottom) with $r_{-2} = 0.275$ [dashed line in (a)]. Inset: interface density voter-like decay at the critical point (system size $2048^2$). (c), (d) $m(t)$ (c) and $\rho(t)$ (d) from ric with various $m_0$ at the critical point described in (b) (system size $16\,384^2$).

Similar investigations have been undertaken in various planes of the four-parameter space, always yielding the same results. This suggests to conjecture that all critical $Z_2$-symmetric rules without bulk noise form a codimension-1 "voter-like" manifold separating order from disorder, characterized by the logarithmic decay of both $\rho$ and $m$.

Does the above picture extend to nonconserving, asymmetric rules without bulk noise? In this case, even though two absorbing states exist, their dynamical roles are not symmetric, and such order-disorder transitions are known [15,16] to belong generically to the universality class of directed percolation (DP). We have indeed observed [9] that for such rules both $m$ and $\rho$ scale at criticality with the *same* exponent $\beta^{\rm DP}/\nu_\parallel^{\rm DP} = 0.450(1)$ [16].

Having specified the required conditions to observe voter-like coarsening, we now turn to the investigation of the scaling laws associated with voter-like points and to an assessment of their universality. To our knowledge, there has been only one attempt at determining the scaling exponents governing the approach to the usual VM. Within the reduced $(p_i, p_b)$ parameter plane defined above, the authors of [12] noted first that increasing $p_i$ along the zero-bulk-noise line $p_b = 0$, up to the VM point $(p_i = \frac{1}{4})$, the (static) magnetization $m_s$ jumps from 0 to $\pm 1$. Thus, the order parameter exponent $\beta = 0$, and, for such a first-order $d = 2$ nonequilibrium critical point, the susceptibility and the correlation length exponents should satisfy $\gamma = 2\nu$. A finite-size scaling analysis of the fluctuations of $m_s$ as $\varepsilon \equiv p_i - \frac{1}{4} \to 0^-$ confirmed the latter relation, and gave $\gamma \approx 1.25$. However, repeating these simulations with much better statistics, we evidenced [9] a systematic decrease of the local exponent $\gamma$ as $\varepsilon \to 0^-$. On general grounds, corrections to scaling in this problem are expected to be logarithmic [17] and, thus, the above $\gamma$ value must be taken cautiously. In fact, a more reliable approach to the VM is from the ordered side $(\varepsilon \to 0^+)$, along the zero-bulk noise line.

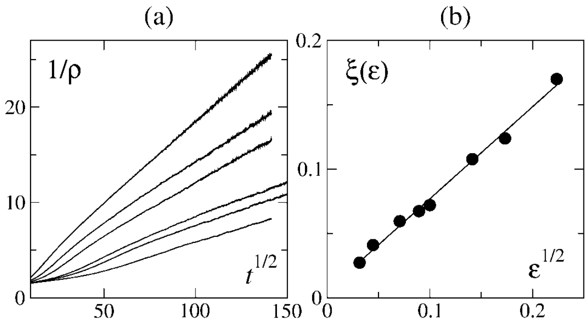


FIG. 4. Phase ordering from symmetric ric approaching the VM from the ordered side (system size $8192^2$). (a) $\rho$ vs $\sqrt{t}$ for various $\varepsilon$ between $10^{-3}$ and $5 \times 10^{-2}$ (from bottom to top); (b) extracted correlation length $\xi = \xi(\varepsilon)$ vs $\sqrt{\varepsilon}$.

The curvature-driven-growth regimes $\rho \sim \xi(\varepsilon)/L(t) \sim \varepsilon^{-\nu}/t^{1/2}$, which eventually settle then in large enough systems, are less prone to the logarithmic corrections mentioned above. The scaling of the correlation length $\xi(\varepsilon)$ yields $\nu = 0.45(7)$, compatible with the simple "diffusive" value $\nu = \frac{1}{2}$ (Fig. 4).

Compounding the above results, a conservative interpretation of the data is that $\beta = 0$, $\gamma = 1$, and $\nu = \frac{1}{2}$, with appropriate logarithmic corrections to scaling. This is further confirmed by similar, preliminary, results obtained when approaching any of the voter-like models studied above [9]. We note that these exponents would obey all the standard scaling relations valid at a critical point, such as $\beta = [d - \lambda(H)]/\lambda(T_i)$, $\gamma = [2\lambda(H) - d]/\lambda(T_i)$, or $\nu = 1/\lambda(T_i)$, with (in $d = 2$) $\lambda(T_i) = \lambda(H) = 2$. We tentatively associate the former RG eigenvalue with the relevance of interfacial noise, and the latter with the presence of a "dynamically self-induced" magnetic field breaking the $Z_2$ symmetry (as happens in KKE-like models). The behavior of the classes of voter-like models we have defined is consistent with such a set of exponents, which would thus define (in $d = 2$) the genuine *voter universality class*. For instance, converting the static critical behavior $m \sim \varepsilon^{\beta}$ to a dynamical one via $\xi \sim \varepsilon^{-\nu} \sim L(t) \sim t^{1/z}$ gives $m \sim t^{-\beta/\nu z}$. Now, if $\beta = 0$ strictly, this implies $m(t) = \mathrm{Cst}$, while if $\beta = 0^+$ is interpreted, as is customary, including logarithmic corrections, then $m(t) \sim 1/\ln t$. Finally, despite the presence of the *a priori* relevant eigenvalue $\lambda(H) = 2$, $m$-conserving rules without $Z_2$ symmetry may still display voter-like growth, because the critical exponent $\delta$ (defined by $m_s \sim H^{1/\delta}$) would read $1/\delta = -1 + d/\lambda(H) = 0$.

Naturally, the RG picture sketched above needs to be substantiated by the study of an appropriate field theory [18], an endeavor left for future work. In particular, one would like to have a better understanding of the RG flow around the voter critical point as the space dimension varies, and of the special role played by $d = 2$, which appears as the *upper critical dimension* where voter-like phase ordering can be induced by the sole presence of interfacial noise [9]. At any rate, we believe the results presented here reveal that the $d = 2$ voter model, which was, up to now, perceived as a marginal system, embodies, in fact, a broad class of models for which coarsening occurs without surface tension. Our findings also suggest that the critical behavior associated with this class of systems is perhaps best characterized as an order-disorder transition driven by the interfacial noise between two absorbing states possessing equivalent "dynamical roles," this symmetry being enforced either by $Z_2$ symmetry of the local rules, or by the global conservation of the magnetization.

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